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LETTER TO THE EDITOR

Comments on some recent multisoliton solutions

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Abstract. It is shown that some recently proposed multisoliton solutions for the nonlinear Klein–Gordon equations can be reduced to a simple form which can be obtained immediately from the equation.

In a series of papers (Burt 1978, Gibbon *et al* 1978, and their earlier references) a procedure is described for finding 'multisoliton-like' solutions of

$$\partial^2\phi/\partial t^2 - \partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2 = F(\phi). \tag{1}$$

The proposal is to take $\phi = \phi(g)$ and choose g to satisfy

$$g_{tt} - g_{xx} - g_{yy} = -g \tag{2}$$

$$g_t^2 - g_x^2 - g_y^2 = -g^2; \tag{3}$$

then

$$g^2\, \mathrm{d}^2g/\mathrm{d}g^2 + g\, \mathrm{d}\phi/\mathrm{d}g + F(\phi) = 0. \tag{4}$$

The single one-dimensional soliton takes this form with

$$g = \exp[(x - Ut)/(1 - U^2)^{1/2}]. \tag{5}$$

The shape of the soliton is determined by (4). For example, in the case

$$F(\phi) = -\phi + 2\phi^3 \tag{6}$$

the single soliton is

$$\phi = 2g/(1 + g^2) = \mathrm{sech}[(x - Ut)/(1 - U^2)^{1/2}]. \tag{7}$$

The idea is to retain the soliton shape provided by (4), but take more general solutions of (2) and (3) for g.

The proposed solutions for g are

$$g = \sum_{i=1}^{N} \exp\theta_i, \qquad \theta_i = p_i x + q_i y - \omega_i t + \delta_i. \tag{8}$$

Hence, from (2) and (3),

$$p_i^2 + q_i^2 = 1 + \omega_i^2 \tag{9}$$

$$p_i p_j + q_i q_j = 1 + \omega_i\omega_j. \tag{10}$$

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However, it is shown below that all such solutions represent patterns moving with constant velocity, and with speed 1, the propagation speed in equation (1). Accordingly, they can be obtained very simply, and in fact more generally, directly from (1). Choose axes x', y' with y' in the direction of translation of the pattern. Then, obviously

$$\phi = \Phi(x', y' - t) \tag{11}$$

is a solution of (1) provided

$$\Phi_{x'x'} + F(\Phi) = 0. \tag{12}$$

This is the single soliton for the dependence of Φ on x'. Any arbitrary parameters in that solution can now be taken as functions of $y' - t$. In the example (6), the solution is

$$\Phi = \operatorname{sech}(x' - a(y' - t)) \tag{13}$$

where a is any function of $y' - t$. It represents an arbitrary waveshape propagating with speed 1 along a sech-shaped hump. From this point of view these solutions do not appear to be very deep.

It might be noted that (7) is a pattern moving with speed 1 in the direction $\{(1 - U^2)^{1/2}, U\}$, and it is recovered from (13) by taking $a = -U(1 - U^2)^{-1/2}(y' - t)$ and rotating the coordinates appropriately.

To prove the above assertion about the solutions obtained from (8)–(10), let $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ denote the 2-vectors

$$\boldsymbol{v}_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \qquad \boldsymbol{w}_i = \begin{pmatrix} 1 \\ \omega_i \end{pmatrix}.$$

Then, from (9) and (10),

$$\boldsymbol{v}_i^2 = \boldsymbol{w}_i^2, \qquad \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = \boldsymbol{w}_i^{\mathrm{T}} \boldsymbol{w}_j. \tag{14}$$

Corresponding vectors $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ have the same length, and corresponding pairs $\boldsymbol{v}_i$, $\boldsymbol{v}_j$ and $\boldsymbol{w}_i$, $\boldsymbol{w}_j$ have the same angle between them. Thus the sets $\boldsymbol{v}_i$ and $\boldsymbol{w}_i$ differ from each other only by a rotation plus a possible reflection. Therefore there exists an orthogonal matrix R such that

$$\boldsymbol{w}_i = R\boldsymbol{v}_i. \tag{15}$$

That is, there exist α, β with

$$\alpha^2 + \beta^2 = 1 \tag{16}$$

such that either

$$1 = -\beta p_i + \alpha q_i \tag{17}$$

$$\omega_i = \alpha p_i + \beta q_i \tag{18}$$

or

$$1 = \beta p_i - \alpha q_i \tag{19}$$

$$\omega_i = \alpha p_i + \beta q_i. \tag{20}$$

In either case (18), (20) show that g, and hence ϕ, are functions of $x - \alpha t$, $y - \beta t$. The pattern moves with constant velocity (α, β), and from (16) the speed is 1. Therefore a change of coordinates leads to (11).

A neater form of the argument is to introduce (15) directly into (8). With $\boldsymbol{\xi}$ denoting the column vector (x, y) and superscripts T denoting transposes, we have

$$g = \sum \exp(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{v}_i - \omega_i t + \delta_i) = \sum \exp(\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{w}_i - \omega_i t + \delta_i). \tag{21}$$

If the transformation $\boldsymbol{\xi}' = \boldsymbol{R}\boldsymbol{\xi}$ is now introduced, we have

$$g = \sum \exp(\boldsymbol{\xi}'^{\mathrm{T}} \cdot \boldsymbol{w}_i - \omega_i t + \delta_i) = \sum \exp(x' + \omega_i(y' - t) + \delta_i) = \mathrm{e}^{x'} G(y' - t). \tag{22}$$

Then (11) follows.

It might be noted that the example in figure 1 of Gibbon *et al* (1978) shows a wave formed by three segments moving in the y' direction with speed 1 in agreement with (11). But there is in fact no reason to restrict the shape to three segments.

In the extension to three space dimensions, with

$$\theta_i = p_i x + q_i y + r_i z - \omega_i t + \delta_i \tag{23}$$

in (8), the restrictions are

$$p_i^2 + q_i^2 + r_i^2 = 1 + \omega_i^2 \tag{24}$$

$$p_i p_j + q_i q_j + r_i r_j = 1 + \omega_i \omega_j. \tag{25}$$

If we introduce 3-vectors

$$\boldsymbol{v}_i = (p_i, q_i, r_i), \qquad \boldsymbol{w}_i = (1, \omega_i, 0) \tag{26}$$

the restrictions again take the form

$$v_i^{\mathrm{T}} v_j = w_i^{\mathrm{T}} w_j \tag{27}$$

(including $i = j$), and again the two sets can only differ by an orthogonal transformation. We have

$$\boldsymbol{w}_i = \boldsymbol{R}\boldsymbol{v}_i. \tag{28}$$

The argument leading to (22) goes through exactly as before; the dependence on z' drops out in the final step, since the third component of $\boldsymbol{w}_i$ is zero. Thus we have only solutions (11).

In the papers referenced the authors note that a count of the conditions (27) in d space dimensions gives $N(N+1)/2$ conditions for the $(d+1)N$ parameters, and that the system may be overdetermined when $N \geqslant 2d+1$. This is not so, however. The relations are satisfied by the orthogonal transformation (28) for any N.

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References

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